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Paretian Poisson Processes

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Abstract Many random populations can be modeled as a countable set of points scattered randomly on the positive half-line. The points may represent magnitudes of earthquakes and tornados, masses of stars, market values of public companies, etc. In this article we explore a specific class of random such populations we coin '*Paretian Poisson processes*'. This class is elemental in statistical physics—connecting together, in a deep and fundamental way, diverse issues including: the Poisson distribution of the Law of Small Numbers; Paretian tail statistics; the Fréchet distribution of Extreme Value Theory; the one-sided Lévy distribution of the Central Limit Theorem; scale-invariance, renormalization and fractality; resilience to random perturbations.

Keywords Probability theory · Stochastic processes · Statistics · Fractals

1 Introduction

Pareto's law—a statistical law describing a power-law connection between positive-valued measurements and their occurrence frequencies—is abundant across the Sciences. First discovered by the Italian economist Vilfredo Pareto at the close of the nineteenth century [1], Pareto's law turned out to be ubiquitous. It is observed in empirical data spanning from human income to earthquake magnitudes and to moon-crater diameters, and various Paretian 'generating mechanisms' were proposed. Excellent reviews of Pareto's law are Chap. 14 in [2], and [3, 4]. Pareto's law, however, is merely the 'tip of the iceberg'—the iceberg itself being the class of *Paretian Poisson processes*, which is the subject of this article.

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Poisson processes constitute the statistical model for the random scattering of points in general domains [5], and have a wide spectrum of applications ranging from Shot Noise [6] to Queueing Systems [7] and to Insurance and Finance [8]. Often, the random scattering of points takes place on the positive half-line: magnitudes (earthquakes, tornados, solar flares); sizes (human income, moon-crater diameters, insurance claims); enumerations (web hits, article citations, city populations). Hereafter, we refer to a Poisson process as *Paretian* if it is defined on the positive half-line and is governed by a decreasing *power-law* rate function (the precise mathematical definition is given in Sect. 4 below).

Paretian Poisson processes are of great importance and centrality in Statistical Physics. Yet, despite their focal role, Paretian Poisson processes are grossly underexposed to the wide physical audience. The aim of this article is to introduce the readers to this important class of processes, and to present a panoramic overview of their key properties:

- Paretian Poisson processes emerge from a 'meta', infinite-dimensional, stochastic limit-law asserting that they are the only possible *stochastic scaling limits* of random populations of independent and identically distributed positive-valued random variables.
- The following, well-known, stochastic limit-laws are one-dimensional projections of the aforementioned 'meta' stochastic limit-law: the *Poisson* distribution of the *Law of Small Numbers*; *Paretian* tail statistics; the *Fréchet* distribution of *Extreme Value Theory*; the one-sided *Lévy* distribution of the *Central Limit Theorem*.
- Paretian Poisson processes constitute the class of *fractal* Poisson processes defined on the positive half-line—fractality being defined either via the notion of *scale-invariance* or via the notion of *renormalization*.
- Paretian Poisson processes constitute the class of Poisson processes defined on the positive half-line which are *resilient* to the action of arbitrary *random multiplicative perturbations*.

In addition, in this article we provide:

- Simple and efficient algorithms for both the *computer simulation* and the *statistical inference* of Paretian Poisson processes.

The reminder of the article is organized as follows. We begin, in Sect. 2, with a simple and generic random population model which gives rise to four stochastic limit-laws: Poisson, Pareto, Fréchet and one-sided Lévy probability distributions. Section 3 reviews the notion of Poisson processes, and the distribution of their exceedances, maxima, and aggregates. Section 4 presents the 'meta' stochastic limit-law and its connection to the four stochastic limit-laws of Sect. 2. Section 5 employs an order-statistics approach to address the computer simulation and the statistical inference of Paretian Poisson processes. Section 6 explores the fractal nature Paretian Poisson processes via scale-invariance and renormalization. We conclude, in Sect. 7, with the resilience of Paretian Poisson processes to the action of random multiplicative perturbations.

2 Stochastic Limit-Laws

Let $\{\xi_1, \xi_2, \ldots\}$ be an arbitrary sequence of Independent and Identically Distributed (IID) positive-valued random variables, governed by the common survival probability function $\mathbf{P}_>(x) = \text{Prob}(\xi_1 > x)$ $(x \ge 0)$.

Given the IID sequence consider the random population

$$\mathcal{P}(\epsilon; n) = \{\epsilon\xi_1, \epsilon\xi_2, \ldots, \epsilon\xi_n\}, \tag{1}$$

where ϵ is a small scale parameter ($\epsilon \ll 1$) and where n is a large size parameter ($n \gg 1$). The random population $\mathcal{P}(\epsilon; n)$ represents a '*macroscopic picture*' of a large set of random events quantified by a positive valued measure.

Probabilistic limit theorems seek stochastic limit-laws attained by the random population $\mathcal{P}(\epsilon; n)$ as $\epsilon \to 0$ and $n \to \infty$. The two most celebrated and well-known examples are: (i) *Extreme Value Theory* which studies the asymptotic behavior of the population-maximum [9–11]; (ii) the *Central Limit Theorem* which studies the asymptotic behavior of the population-aggregate [12–14].

In order to ensure the existence of stochastic limit-laws two key questions need be addressed: (i) what condition should the survival probability function $\mathbf{P}_>(\cdot)$ satisfy? and, (ii) how should the parameters ϵ and n scale jointly as $\epsilon \to 0$ and $n \to \infty$?

The answer to the first question involves the notion of *regular variation* [15]. Regularly varying functions are generalizations of asymptotic power-laws. A real function $f(\cdot)$ is said to be regularly varying at infinity if the limit $\lim_{x\to\infty} f(yx)/f(x)$ exists for all positive constants y. Theory shows that if the function $f(\cdot)$ is regularly varying then $\lim_{x\to\infty} f(yx)/f(x) = y^\nu$, where the exponent ν is a real parameter called the "exponent of regular variation".

The following pair of conditions—the first regarding the survival probability function $\mathbf{P}_>(\cdot)$, the second regarding the parameters ϵ and n—are necessary and sufficient in order to ensure the existence of stochastic limit-laws:

Condition 1 *The survival probability function* $\mathbf{P}_>(\cdot)$ *is regularly varying at infinity*:

$$\lim_{x\to\infty} \frac{\mathbf{P}_>(yx)}{\mathbf{P}_>(x)} = y^{-\alpha} \quad (y > 0), \tag{2}$$

where α *is an arbitrary positive exponent.*

Condition 2 *The population size* n *is of the order of* $1/\mathbf{P}_>(1/\epsilon)$:

$$\lim_{\substack{\epsilon\to 0\\ n\to\infty}} n\mathbf{P}_>(1/\epsilon) = c, \tag{3}$$

where c *is an arbitrary positive constant.*

We now turn to describe four different stochastic limit-laws attainable from the random population $\mathcal{P}(\epsilon; n)$: *Poisson*, *Pareto*, *Fréchet* and one-sided *Lévy*. These stochastic limit-laws are classic results in Probability Theory—the first three fall under the category of *Extreme Value Theory*, and the fourth falls under the category of the *Central Limit Theorem*.

Below let the set $\mathcal{P}_l(\epsilon; n) = \mathcal{P}(\epsilon; n) \cap (l, \infty)$ denote the sub-population consisting of the population points residing above the positive level l. And, henceforth, let the shorthand *lim* denote the double limit $\epsilon \to 0$ and $n \to \infty$.

2.1 Poisson

Consider the number of points $N_{\mathcal{P}(\epsilon;n)}(l)$ of the population $\mathcal{P}(\epsilon; n)$ which exceed the positive threshold level l. The random variable $N_{\mathcal{P}(\epsilon;n)}(l)$ equals the size of the sub-population $\mathcal{P}_l(\epsilon; n)$. The stochastic limit $N(l) = \lim N_{\mathcal{P}(\epsilon;n)}(l)$ exists if and only if Conditions 1 and 2 are met—in which case it is *Poisson distributed* with mean

$$\langle N(l)\rangle = cl^{-\alpha}. \tag{4}$$

The convergence of 'rare events'—in our case the exceedances of a given threshold—to a limiting Poisson distribution is often referred to as the "*Law of Small Numbers*" ([16], Sects. VI.5 and VI.6).

2.2 Pareto

Consider the size $S_{\mathcal{P}(\epsilon;n)}(l)$ of a population-point exceeding the positive threshold level l. The random variable $S_{\mathcal{P}(\epsilon;n)}(l)$ represents a generic member of the sub-population $\mathcal{P}_l(\epsilon;n)$. The stochastic limit $S(l)=\lim S_{\mathcal{P}(\epsilon;n)}(l)$ exists if and only if Condition 1 is met—in which case it is *Pareto distributed* with survival probability function

$$\mathrm{Prob}\,(S(l)>x)=(x/l)^{-\alpha}\quad (x\geq l). \tag{5}$$

This stochastic limit-law is one simple and general explanation to the widespread appearance of Paretian tail statistics in empirical data [4].

2.3 Fréchet

Consider the maximum $M_{\mathcal{P}(\epsilon;n)}$ of the population $\mathcal{P}(\epsilon;n)$. The random variable $M_{\mathcal{P}(\epsilon;n)}$ equals the size of the population's maximal point. The stochastic limit $M=\lim M_{\mathcal{P}(\epsilon;n)}$ exists if and only if Conditions 1 and 2 are met—in which case it is *Fréchet distributed* with cumulative distribution function

$$\mathrm{Prob}\,(M\leq x)=\exp\left\{-cx^{-\alpha}\right\}\quad (x\geq 0). \tag{6}$$

Extreme Value Theory asserts that the Fréchet distribution is the only possible stochastic scaling limit—supported on the positive half-line—of maxima of IID random variables [9–11].

2.4 Lévy

Consider the aggregate $A_{\mathcal{P}(\epsilon;n)}$ of the population $\mathcal{P}(\epsilon;n)$. The random variable $A_{\mathcal{P}(\epsilon;n)}$ equals the sum of the population points. The stochastic limit $A=\lim A_{\mathcal{P}(\epsilon;n)}$ exists if and only if Conditions 1 and 2 are met—provided that the exponent α is in the range $0<\alpha<1$—in which case it is one-sided *Lévy distributed* with Laplace transform

$$\langle\exp\{-\theta A\}\rangle=\exp\{-\Gamma(1-\alpha)c\theta^{\alpha}\}\quad (\theta\geq 0). \tag{7}$$

The Central Limit Theorem asserts that the one-sided Lévy distribution is the only possible stochastic scaling limit—supported on the positive half-line—of aggregates of IID random variables [12–14].

3 Poisson Processes

A Poisson process Π with rate function $\mathbf{r}(t)$ $(t>0)$ is a countable collection of points scattered randomly on the positive half-line, characterized by the following pair of properties [5]: (i) the number of points residing within the interval I is a Poisson-distributed random variable with mean $\int_I \mathbf{r}(t)dt$; and, (ii) the number of points residing within disjoint intervals are independent random variables.

Let us now turn to examine the exceedances, maxima, and aggregates—discussed in the previous section in the context of the random population $\mathcal{P}(\epsilon; n)$—in the context of Poisson processes.

Below, we consider an arbitrary Poisson process Π with rate function $\mathbf{r}(t)$ ($t > 0$) which is integrable at infinity, and let $\Pi_l = \Pi \cap (l, \infty)$ denote the subset of points residing above the positive level l.

3.1 Exceedances

Consider the number of points $N_\Pi(l)$ exceeding the positive threshold level l. The random variable $N_\Pi(l)$ equals the size of the subset Π_l, and is thus Poisson distributed with mean

$$\langle N_\Pi(l) \rangle = \int_l^\infty \mathbf{r}(t)dt. \tag{8}$$

The 'existence theorem' of the Theory of Poisson processes ([5], Sect. 2.5) further implies that: (i) the subset Π_l forms an IID sequence of random variables (which are independent of the subset's size); and, (ii) the size $S_\Pi(l)$ of a generic member of the subset Π_l is governed by the survival probability function

$$\text{Prob}\,(S_\Pi(l) > x) = \frac{\int_x^\infty \mathbf{r}(t)dt}{\int_l^\infty \mathbf{r}(t)dt} \quad (x \geq l). \tag{9}$$

3.2 Maximum

Consider the maximum M_Π of the Poisson process Π—i.e., the maximal point of the random set Π. The maximum M_Π is no larger than the level x if and only if the random variable $N_\Pi(x)$ equals zero: $\{M_\Pi \leq x\} \Leftrightarrow \{N_\Pi(x) = 0\}$. Since the random variable $N_\Pi(x)$ is Poisson distributed with mean $\int_x^\infty \mathbf{r}(t)dt$, we obtain that the maximum M_Π is governed by the cumulative distribution function

$$\text{Prob}\,(M_\Pi \leq x) = \exp\left\{-\int_x^\infty \mathbf{r}(t)dt\right\} \quad (x \geq 0). \tag{10}$$

3.3 Aggregate

Consider the aggregate A_Π of the Poisson process Π—i.e., the sum of points of the random set Π. Campbell's theorem of the theory of Poisson processes ([5], Sect. 3.2) asserts that: (i) the aggregate A_Π is summable if and only if the integral $\int_0^\infty \min\{t, 1\}\mathbf{r}(t)dt$ is convergent; and, (ii) if summable then the Laplace transform of the aggregate is given by

$$\langle \exp\{-\theta A_\Pi\} \rangle = \exp\left\{-\int_0^\infty (1 - \exp\{-\theta t\})\,\mathbf{r}(t)dt\right\} \quad (\theta \geq 0) \tag{11}$$

4 Paretian Poisson Processes

Comparing, respectively, equations (4)–(7) of Sect. 2 to (8)–(11) of Sect. 3, we arrive at the following conclusion:

Conclusion 3 *The limiting random variables* $N(l), S(l), M, A$ *of Sect.* 2 *are equal—respectively and in law—to the random variables* $N_\Pi(l), S_\Pi(l), M_\Pi, A_\Pi$ *of Sect.* 3 *if and only if the rate function of the Poisson process* Π *is given by*

$$\mathbf{r}(t) = c\alpha t^{-(1+\alpha)} \quad (t > 0). \tag{12}$$

Henceforth, we refer to a Poisson process defined on the positive half-line as *Paretian* if its rate function admits the power-law form of (12)—the parameter α being its *Paretian exponent*, and the parameter c being its *Paretian amplitude*.

Conclusion 3 asserts that Paretian Poissonian processes underlie all four stochastic limit-laws discussed in Sect. 2: Poisson, Pareto, Fréchet and one-sided Lévy. In fact, the stochastic limit-laws of the Sect. 2 are merely *one-dimensional projections* of the following, *infinite-dimensional*, *'meta'* stochastic limit-law:

Proposition 4 *The stochastic limit* $\mathcal{P} = \lim \mathcal{P}(\epsilon; n)$ *exists if and only if Conditions* 1 *and* 2 *are met—in which case the limiting population* $\mathcal{P}$ *is a Paretian Poisson process with exponent* α *and amplitude* c.

Proposition 4, whose proof is given in the Appendix 9.1, is a special case of general statistical limit theorems referred to in Probability Theory as "the convergence of empirical measures to limiting Poisson random measures" ([17], Theorems 10 and 11; [18]).

We emphasize that not all functionals of the random population $\mathcal{P}(\epsilon; n)$ converge stochastically to the corresponding functionals of the limiting population $\mathcal{P}$. An example of an important population functional—the Lorenz curve [19]—displaying a *discontinuous* behavior at the limit $\mathcal{P} = \lim \mathcal{P}(\epsilon; n)$ is given in the Appendix 9.2.

4.1 Universality

The stochastic limit-law of Proposition 4, as well as its one-dimensional projections—the four stochastic limit-laws of Sect. 2, turn out to be independent of the details of the underlying survival probability function $\mathbf{P}_>(\cdot)$. In the stochastic limit ($\epsilon \to 0$ and $n \to \infty$) the entire functional structure of the survival probability function $\mathbf{P}_>(\cdot)$ collapses to one single parameter: the exponent α.

The random variables $\{\xi_1, \xi_2, \ldots\}$, governed by the survival probability function $\mathbf{P}_>(\cdot)$, represent the population at its *microscopic* level. The stochastic limit $\mathcal{P}$, on the other hand, represent the population at its *macroscopic* level. The only information communicated from the micro-level to the macro-level—and determining the structure of the macro-level—is the exponent α. At the micro-level the parameter α is the exponent of regular variation (of the survival probability function $\mathbf{P}_>(\cdot)$), and at the macro-level the parameter α is the Paretian exponent.

This 'collapse of information' taking place in the transition from the micro-level to the macro-level renders the aforementioned stochastic limit-laws *universal*.

4.2 Macroscopic Observability

The ubiquity of Pareto's law in diverse 'real world' systems motivated researchers to seek 'universal mechanisms' capable of generating this law. Examples include Preferential Attachment (Yule process [20], Simon's model [21]), Self-Organized Criticality ([22] and references therein), and the Oligarchy Mechanism (for the universal generation of Pareto's law with integer-valued exponents [23]).

Equation (5) and Proposition 4 provide a universal *statistical explanation*—rather than a universal *generating mechanism*—for the emergence of power-laws in 'real world' statistical data. In the context of large data sets drawn randomly and independently from an arbitrary positive-valued probability distribution we asked: "What statistical regularities are *macroscopically observable*?". The answer, given by Proposition 4, was: "Paretian Poisson processes". And, most important, the precondition for the emergence of a macroscopically observable statistical regularity is that the survival probability of the underlying distribution be *regularly varying* at infinity.

The situation is as follows. On the microscopic level, the precondition of regular variation may or may not hold. If it holds then, on the macroscopic level, a statistical regularity will exist and it will be 'Paretian'. And if it does not hold then, on the macroscopic level, no statistical regularity will exist. Macroscopically—in the positive-valued IID setting considered—we either see 'Paretian statistical regularity' or see no statistical regularity whatsoever.

As for the microscopic precondition of regular variation—be it ubiquitous or rare—the cases in which it holds are precisely the cases in which a macroscopic statistical regularity exists. In these cases particular and specific 'microscopic models' (as the ones noted above [20–23]) are required in order to explain the existence of regular variation.

In settings which are not IID however, non-Paretian macroscopic statistical regularities may certainly appear. This is well exemplified by the empirical distributions of income in human societies—a result of countless interactions taking place in human economies—which have log-Normal bulks and are Paretian only in their tails [24].

4.3 Phase Transition

Paretian Poisson processes undergo a *phase transition* at the Paretian exponent value $\alpha = 1$. The phase transition affects the statistical behavior of both the exceedance-size $S(l)$ and the aggregate A. Indeed, the exceedance-size $S(l)$ is of infinite mean ($\langle S(l)\rangle = \infty$) in the exponent range $0 < \alpha \le 1$, and is of finite mean ($\langle S(l)\rangle = l\alpha/(\alpha-1)$) in the exponent range $\alpha > 1$. And, the aggregate A is finite (with Laplace transform given by (7)) in the exponent range $0 < \alpha < 1$, and is infinite (with probability one) in the exponent range $\alpha \ge 1$. The phase transition further affects the Lorenz curve [25] and the oligarchic structure [26] of Paretian Poisson processes.

5 Order-Statistics

This Section focuses on the *order-statistics* of the limiting population $\mathcal{P}$—a Paretian Poisson process with exponent α and amplitude c. Since the population $\mathcal{P}$ is infinite, it has an infinite sequence of order-statistics $O_1 > O_2 > O_3 > \cdots$, where O_1 denotes population's maximal point, O_2 denotes population's second-maximal point, etc.

5.1 Simulation

The power-law Poissonian structure of the limiting population $\mathcal{P}$ induces the following stochastic representation for its sequence of order-statistics [27]:[1]

$$O_k \overset{\text{law}}{=} \left(\frac{c}{\mathcal{E}_1 + \cdots + \mathcal{E}_k}\right)^{\frac{1}{\alpha}} \tag{13}$$

[1] Here and hereafter, the sign $\overset{\text{law}}{=}$ denotes equality *in law* of random entities.

($k = 1, 2, \ldots$), where $\{\mathcal{E}_k\}_{k=1}^{\infty}$ is an IID sequence of exponentially-distributed random variables with unit mean.

Equation (13) confirms with the fact that the order-statistics are *dependent* random variables. What is surprising and unexpected is the fact that the *consecutive ratios* of the order-statistics turn out to be *independent* random variables—admitting the stochastic representation [27]:

$$\frac{O_{k+1}}{O_k} \overset{\text{law}}{=} (U_k)^{\frac{1}{\alpha k}} \tag{14}$$

($k = 1, 2, \ldots$), where $\{U_k\}_{k=1}^{\infty}$ is an IID sequence of random variables distributed uniformly on the unit interval.

The stochastic representations of (13) and (14) provide highly efficient and remarkably simple algorithms for the *computer simulation* of the order-statistics, and of the order-statistics ratios, of Paretian Poisson processes.

5.2 Statistical inference

In the case of vast populations its often so that the *observed data* consists of the top-n order-statistics $O_1 > \cdots > O_n$—rather than being a standard random sample of the population.

Having observed the top-n sample $\{O_k\}_{k=1}^{n}$ of a Poisson process (defined on the positive half-line), two key questions are of interest: (i) Is the underlying Poisson process *Paretian*? (ii) If the answer to the first question is affirmative, then what is the value of the *Paretian exponent* α?

The top-n sample $\{O_k\}_{k=1}^{n}$ however, is *not* an IID sample. Hence, standard statistical methods for IID samples *cannot* be applied to the top-n sample $\{O_k\}_{k=1}^{n}$ in order to infer the statistics of the underlying Poisson process. Yet, under the null hypothesis that the underlying Poisson process is Paretian, (14) implies that the random variables

$$V_k = k \ln\left(\frac{O_k}{O_{k+1}}\right) \tag{15}$$

($k = 1, \ldots, n-1$) form an IID sequence of exponentially-distributed random variables with mean $1/\alpha$. Thus, conducting a standard goodness-of-fit statistical test to the sample $\{V_k\}_{k=1}^{n-1}$ we can conclude whether to accept or reject the null hypothesis.

Having accepted the null hypothesis, we set

$$\Theta_n = \sum_{k=1}^{n-1} \ln\left(\frac{O_k}{O_n}\right), \tag{16}$$

and estimate the Paretian exponent α according to the following proposition:

Proposition 5

1. *The statistic* n/Θ_n *is a Maximal Likelihood Estimator of the Paretian exponent* α, *and its probability density function is unimodal with mode* α *and peak* $\sim \sqrt{n}/(\alpha\sqrt{2\pi})$ *(as* $n \to \infty$*).*
2. *The statistic* $(n-2)/\Theta_n$ *is an Unbiased Estimator of the Paretian exponent* α, *and its variance equals* $\alpha^2/(n-3)$.

3. *The skew and the kurtosis of both statistics* n/Θ_n *and* $(n-2)/\Theta_n$ *are given by*:

$$Skew = 4\frac{\sqrt{n-3}}{(n-4)}, \qquad Kurtosis = 3\frac{(n-3)(n+4)}{(n-4)(n-5)}. \tag{17}$$

The proof of Proposition 5 is given in the Appendix 9.3. This proof also implies that the statistic $\Theta_n/(n-1)$ is an Unbiased Estimator of the parameter $1/\alpha$, and that it converges in probability (as $n \to \infty$) to its mean $1/\alpha$. Moreover, the Theory of Large Deviations (see, for example, [28]) ensures that the convergence in probability takes place at an exponential pace. (These assertions are consequences of the stochastic representation of (41), appearing in the proof of Proposition 5.)

Proposition 7 in [25] asserts that if the rate function $\mathbf{r}(t)$ $(t > 0)$ of the underlying Poisson process is regularly varying *at the origin* with exponent $\nu = -(1+\alpha)$ then the statistic n/Θ_n converges with probability one (as $n \to \infty$) to the Paretian exponent α.

We note that the statistic $(n-1)/\Theta_n$ is the *Hill Estimator* [29]—the Maximal Likelihood Estimator of the Paretian exponent α, based on the top-n sample $\{O_k\}_{k=1}^n$, in the case of finite IID populations drawn from a *Pareto distribution*.

6 Fractality

This Section describes two different definitions of *fractality* in the context of Poisson processes defined on the positive half-line—one based on the notion of *scale invariance*, and the other based on the notion of *renormalization*. As shall be demonstrated, both these definitions identify fractality with the class of Paretian Poisson processes.

For a comprehensive study of fractality in the context of Poisson processes, as well as for the detailed proofs of the results stated in this Section, the readers are referred to [30]. Below, we consider an arbitrary Poisson process Π with rate function $\mathbf{r}(t)$ $(t > 0)$ which is integrable at infinity.

6.1 Scale Invariance

Consider the subset $\Pi_l = \Pi \cap (l, \infty)$ consisting of the points of the process Π residing above the positive level l. As noted in Sect. 3 above, the subset Π_l forms an IID sequence of random variables with generic size $S_\Pi(l)$ (whose survival probability function is given by (9)).

If we rescale the generic size $S_\Pi(l)$ with respect to level l we obtain the rescaled generic size $\widehat{S_\Pi}(l) = S_\Pi(l)/l$—which is governed by the survival probability function

$$\text{Prob}\left(\widehat{S_\Pi}(l) > x\right) = \frac{\int_{lx}^{\infty} \mathbf{r}(t)dt}{\int_{l}^{\infty} \mathbf{r}(t)dt} \quad (x \geq 1). \tag{18}$$

The Poisson process Π is defined *scale-invariant* if its rescaled generic sizes $\{\widehat{S_\Pi}(l)\}_{l>0}$ are independent of the level parameter l [30].

Equation (18) implies that the Poisson process Π is scale-invariant if and only if its rate function is a *power-law*. Hence, the class of *scale-invariant* Poisson processes coincides with the class of *Paretian* Poisson processes.

6.2 Renormalization

Consider the p-order renormalization $\mathcal{R}_p(\Pi)$ of the process Π defined as follows (p being a positive parameter): (i) replace the Poisson process Π with a Poisson process Π_p governed by the rate function $\mathbf{r}_p(t) = p\mathbf{r}(t)$; (ii) rescale the points of the Poisson process Π_p by the factor $\sigma(p)$. The resulting p-order renormalization is given by the set

$$\mathcal{R}_p(\Pi) = \left\{ \frac{1}{\sigma(p)} \cdot \pi \right\}_{\pi \in \Pi_p} . \tag{19}$$

The renormalization is required to be *consistent*: A p-order renormalization followed by a q-order renormalization should equal, in law, a pq-order renormalization. This consistency requirement implies that the renormalization function $\sigma(\cdot)$ need be a power-law [30]: $\sigma(p) = p^\beta$ where β is an arbitrary positive exponent.

The 'displacement theorem' of the Theory of Poisson processes ([5], Sect. 5.5) further implies that the p-order renormalization $\mathcal{R}_p(\Pi)$ is a Poisson process with rate function

$$\left(\mathcal{R}_p(\mathbf{r})\right)(t) = p^{1+\beta}\mathbf{r}\left(p^\beta t\right) \quad (t > 0). \tag{20}$$

The Poisson process Π is defined a *renormalization fixed-point* if the renormalizations $\{\mathcal{R}_p\}_{p>0}$ leave it statistically unchanged [30]: $\mathcal{R}_p(\Pi) \stackrel{\text{law}}{=} \Pi$ for all $p > 0$.

Equation (20) implies that the Poisson process Π is a renormalization fixed-point if and only if its rate function is a *power-law*. Hence, the class of *renormalization fixed-point* Poisson processes coincides with the class of *Paretian* Poisson processes.

The motivation for having constructed and defined the Poissonian renormalizations $\{\mathcal{R}_p\}_{p>0}$ as presented above is explained in the Appendix 9.4.

7 Perturbations

In this last Section we explore the statistical deformation of Paretian Poisson processes under the action of arbitrary *random multiplicative perturbations*.

Consider a random shock applied to a population represented by an arbitrary countable collection of positive-valued points $\Omega = \{\omega_k\}_k$. The shock perturbs each point of the population—independently of all other points—by the random positive factor ζ. Hence, the population is perturbed from its 'pre-shock' position $\Omega = \{\omega_k\}_k$ to the 'post-shock' position $\Upsilon_\zeta(\Omega) = \{\zeta_k\omega_k\}_k$, where $\{\zeta_k\}_k$ are IID copies of the random variable ζ. We coin the set $\Upsilon_\zeta(\Omega)$ the ζ*-perturbation* of the set Ω.

Consider now an arbitrary Poisson process Π with rate function $\mathbf{r}(t)$ $(t > 0)$ which is integrable at infinity. The 'displacement theorem' of the Theory of Poisson processes ([5], Sect. 5.5) implies that the ζ-perturbation $\Upsilon_\zeta(\Pi)$ of the process Π is a Poisson process with rate function

$$\left(\Upsilon_\zeta(\mathbf{r})\right)(t) = \int_0^\infty \mathbf{r}\left(\frac{t}{s}\right)\frac{1}{s}F_\zeta(ds) \quad (t > 0), \tag{21}$$

where $F_\zeta(\cdot)$ is the cumulative distribution function of the random variable ζ.

If the Poisson process under consideration is *Paretian* then (21) yields

$$\left(\Upsilon_\zeta(\mathbf{r})\right)(t) = \langle \zeta^\alpha \rangle\, \mathbf{r}(t) \quad (t > 0), \tag{22}$$

where α is the Paretian exponent. Namely, ζ-perturbations map the class of Paretian Poisson process onto itself as follows: (i) the Paretian exponent α is left unchanged; and, (ii) the Paretian amplitude is multiplied by the factor $\langle \zeta^{\alpha} \rangle$. Up to their amplitudes, Paretian Poisson processes are left *invariant* under the action of random multiplicative perturbations.

In fact, (22) is unique to Paretian Poisson processes and *characterizes* them: A Poisson process is Paretian with exponent α if and only if (22) holds for all ζ-perturbations. (The 'only if' part of this assertion follows by considering deterministic ζ-perturbations.)

More generally, we define the Poisson process Π *resilient* to the action of random multiplicative perturbations if

$$\left(\Upsilon_{\zeta}(\mathbf{r})\right)(t) = \Lambda(\zeta)\mathbf{r}(t) \quad (t > 0) \tag{23}$$

holds for all ζ-perturbations, where $\Lambda(\zeta)$ is a some positive-valued functional of the random perturbation factor ζ. Namely, a Poisson process Π is *resilient* if—up to a multiplicative factor—its rate function $\mathbf{r}(t)$ $(t > 0)$ is left *invariant* under the action of random multiplicative perturbations.

The following Proposition, whose proof is given in the Appendix 9.5, asserts that the class of *resilient* Poisson processes coincides with the class of *Paretian* Poisson processes:

Proposition 6 *A Poisson process* Π *is resilient if and only if it is Paretian.*

8 Conclusions

This article explored the class of *Paretian Poisson processes*—Poisson processes defined on the positive half-line and governed by decreasing power-law rate functions. As was demonstrated, the class of Paretian Poisson processes is central in statistical physics: It connects together elemental issues—which, at first glance, seem unrelated—via a deep and fundamental underlying statistical structure.

The Poisson distribution of the Law of Small Numbers, Paretian tail statistics, the Fréchet distribution of Extreme Value Theory, the one-sided Lévy distribution of the Central Limit Theorem—are all one-dimensional projections of an underlying infinite-dimensional 'meta' stochastic limit-law which asserts that Paretian Poisson processes are the only possible stochastic scaling limits of random populations of IID positive-valued random variables.

Moreover, within the totality of all Poisson processes defined on the positive half-line, Paretian Poisson processes constitutes the class of processes which are: (i) invariant under changes of scale; (ii) invariant under renormalizations; (iii) resilient to the action of random multiplicative perturbations. The first two features render Paretian Poisson processes fractal, while the third renders their statistical structure highly stable.

We hope that this article will encourage Scientists—when encountering and investigating random populations—to bear the 'picture' of Paretian Poisson processes in mind. Being aware of the entire 'Pareto-Poisson iceberg', rather than merely of its 'Paretian tip-of-the-iceberg', may very well lead to deeper an understanding and insight of the populations under investigation.

Appendix

9.1 Proof of Proposition 4

A general point process Ω defined on the positive half-line is an arbitrary countable collection of points scattered randomly on the positive half-line. The *generating function* of a general point process Ω is given by

$$G_{\Omega}(\phi) = \left\langle \prod_{\omega\in\Omega} \phi(\omega) \right\rangle, \tag{24}$$

where the variable $\phi(\cdot)$ is a smooth test function which takes values in the unit interval and satisfies $\phi(0) = 1$.

Throughout the proof the shorthand *lim* denotes the double limit $\epsilon \to 0$ and $n \to \infty$.

The generating function of the random population $\mathcal{P}(\epsilon; n)$ of (1) is given by

$$G_{\mathcal{P}(\epsilon;n)}(\phi) = \left\langle \prod_{k=1}^{n} \phi(\epsilon\xi_k) \right\rangle = \langle \phi(\epsilon\xi_1)\rangle^n . \tag{25}$$

Noting that

$$\begin{aligned} \langle \phi(\epsilon\xi_1)\rangle &= \phi(0) + \int_0^\infty \phi'(x)\mathrm{Prob}\,(\epsilon\xi_1 > x)\,dx \\ &= 1 + \int_0^\infty \phi'(x)\mathbf{P}_>(x/\epsilon)dx, \end{aligned} \tag{26}$$

we obtain that

$$G_{\mathcal{P}(\epsilon;n)}(\phi) = \left(1 + \int_0^\infty \phi'(x)\mathbf{P}_>(x/\epsilon)dx\right)^n . \tag{27}$$

A non-trivial limit $G(\phi) = \lim G_{\mathcal{P}(\epsilon;n)}(\phi)$ exists if and only if a non-trivial limit

$$I(\phi) = \lim n \int_0^\infty \phi'(x)\mathbf{P}_>(x/\epsilon)dx \tag{28}$$

exists—in which case $G(\phi) = \exp\{I(\phi)\}$.

Noting that

$$n\int_0^\infty \phi'(x)\mathbf{P}_>(x/\epsilon)dx = (n\mathbf{P}_>(1/\epsilon))\int_0^\infty \phi'(x)\left(\frac{\mathbf{P}_>(x/\epsilon)}{\mathbf{P}_>(1/\epsilon)}\right)dx \tag{29}$$

it is straightforward to observe that a non-trivial limit $I(\phi)$ exists if and only if Conditions 1 and 2 are met—in which case

$$\begin{aligned} I(\phi) &= c\int_0^\infty \phi'(x)x^{-\alpha}dx = c\int_0^\infty \phi'(x)\left(\int_x^\infty \alpha t^{-(1+\alpha)}dt\right)dx \\ &= \int_0^\infty \left(\int_0^t \phi'(x)dx\right)\left(c\alpha t^{-(1+\alpha)}\right)dt = \int_0^\infty (\phi(t)-1)\left(c\alpha t^{-(1+\alpha)}\right)dt. \end{aligned} \tag{30}$$

Thus, we conclude that: A non-trivial limit $G(\phi) = \lim G_{\mathcal{P}(\epsilon;n)}(\phi)$ exists if and only if Conditions 1 and 2 are met—in which case

$$G(\phi) = \exp\left\{\int_0^\infty (\phi(t) - 1)\,\mathbf{r}(t)dt\right\} \tag{31}$$

where $\mathbf{r}(t) = c\alpha t^{-(1+\alpha)}$ $(t > 0)$.

The Right-Hand-Side of (31), however, is the *generating function* of a Poisson process $\mathcal{P}$ defined on the positive half-line and governed by the rate function $\mathbf{r}(t)$ $(t > 0)$ [5]. Hence, we arrive at the conclusion of Proposition 4:

The stochastic limit $\mathcal{P} = \lim \mathcal{P}(\epsilon; n)$ exists if and only if Conditions 1 and 2 are met—in which case the stochastic limit $\mathcal{P}$ is a Paretian Poisson process governed by the power-law rate function $\mathbf{r}(t) = c\alpha t^{-(1+\alpha)}$ $(t > 0)$.

9.2 Lorenz Curves

The Lorenz curve is a statistical method of representing distribution functions of different types and ranges on a single universally-calibrated scale. It was devised in 1905 by the American statistician Max Lorenz [19] in order to quantitatively measure, in a universally-calibrated way, the distribution of wealth within human populations—no matter what the population considered is, no matter in what currency wealth is measured, and no matter the range of the population's wealth values.

The Lorenz curve $y = L_\Omega(x)$ of a given population Ω reads out as follows: The top $100x\%$ of the population are in possession of $100y\%$ of the total population's wealth. The Lorenz curve is defined on the unit interval $(0 \le x \le 1)$, ranges over the unit interval $(0 \le y \le 1)$, and is monotone non-decreasing from the point $L_\Omega(0) = 0$ to the point $L_\Omega(1) = 1$.

Let us now turn to examine the Lorenz curve of the random population $\mathcal{P}(\epsilon; n)$ in the double limit $\epsilon \to 0$ and $n \to \infty$, and the Lorenz curve of the limiting population $\mathcal{P} = \lim \mathcal{P}(\epsilon; n)$.

9.2.1 The Population $\mathcal{P}(\epsilon; n)$

A probabilistic limit theorem of the Glivenko-Cantelli type asserts that [31, 32]: If the IID random variables $\{\xi_1, \xi_2, \ldots\}$ possess a finite mean $\langle \xi_1 \rangle = \mu < \infty$ then the limiting Lorenz curve $L(x) = \lim L_{\mathcal{P}(\epsilon;n)}(x)$ of the random population $\mathcal{P}(\epsilon; n)$ exists and is given by

$$L(x) = \frac{1}{\mu}\int_0^x \mathbf{P}_>^{-1}(u)du \tag{32}$$

$(0 \le x \le 1)$, where $\mathbf{P}_>^{-1}(\cdot)$ is the inverse of the survival probability function $\mathbf{P}_>(\cdot)$.

Note that the Lorenzian stochastic limit-law for the random population $\mathcal{P}(\epsilon; n)$ is *not universal*. Indeed, the limiting Lorenz curve $L(x) = \lim L_{\mathcal{P}(\epsilon;n)}(x)$ turns out to be contingent on the specific details of the survival probability function $\mathbf{P}_>(\cdot)$—in sharp contrast to the stochastic limit-laws of Sect. 2.

9.2.2 The Population $\mathcal{P}$

A recent 'Lorenzian analysis' of Paretian Poissonian processes asserts that [25]: If the Paretian exponent α is in the range $\alpha > 1$ then the Lorenz curve of the limiting population $\mathcal{P}$ is given by

$$L_{\mathcal{P}}(x) = x^{1-\frac{1}{\alpha}} \tag{33}$$

($0 \le x \le 1$). (The requirement that the exponent α be in the range $\alpha > 1$, in the case of the limiting population $\mathcal{P}$, is the 'Poissonian analogue' of the finite mean requirement $\mu < \infty$ in the case of the random population $\mathcal{P}(\epsilon; n)$.)

9.2.3 *Discontinuity*

The Lorenz curve displays a *discontinuous* behavior at the double limit $\epsilon \to 0$ and $n \to \infty$. Indeed, the limiting Lorenz curve $L(x) = \lim L_{\mathcal{P}(\epsilon;n)}(x)$, in general, does not coincide with the Lorenz curve $L_{\mathcal{P}}(x)$ of the limiting population $\mathcal{P} = \lim \mathcal{P}(\epsilon; n)$. Namely:

$$\lim L_{\mathcal{P}(\epsilon;n)}(x) \neq L_{\lim \mathcal{P}(\epsilon;n)}(x). \tag{34}$$

This discontinuous behavior is markedly different than the continuous behavior displayed by the population-functionals considered in Sects. 2 and 3—exceedances, maxima, and aggregates.

It is interesting to note that equality holds in (34) if and only if the survival probability function $\mathbf{P}_>(\cdot)$ is *Paretian*: $\mathbf{P}_>(x) = (x/b)^{-\alpha}$ ($x \ge b$), where b is an arbitrary positive lower bound. This assertion is intimately related to the notion of '*Lorenzian fractality*' introduced in [25].

(The 'if' part of this assertion follows from the substitution of the Paretian survival probability $\mathbf{P}_>(x) = (x/b)^{-\alpha}$ into (32); the 'only if' part of the assertion follows from equating equations (32) and (33), and thereafter extracting the survival probability function $\mathbf{P}_>(\cdot)$.)

9.3 Proof of Proposition 5

We split the proof of Proposition 5 into three parts.

9.3.1 *Maximal Likelihood Estimator*

The multidimensional probability density function of the top-n order-statistics $\{O_k\}_{k=1}^n$ of a Poisson process with rate function $\mathbf{r}(t)$ ($t > 0$) is given by [27]:

$$f_n(t_1, \ldots, t_n) = \mathbf{r}(t_1) \cdots \mathbf{r}(t_n) \exp\left\{-\int_{t_n}^{\infty} \mathbf{r}(t)dt\right\} \tag{35}$$

($t_1 > \cdots > t_n > 0$).

Hence, the *log-likelihood* of the top-n order-statistics of a Paretian Poisson process—governed by the power-law rate function $\mathbf{r}(t) = c\alpha t^{-(1+\alpha)}$ ($t > 0$)—is given by:

$$L_n(t_1, \ldots, t_n; c, \alpha) = n \ln(c) + n \ln(\alpha) - (\alpha + 1) \ln(t_1 \cdots t_n) - c\,(t_n)^{-\alpha} \tag{36}$$

($t_1 > \cdots > t_n > 0;\ c, \alpha > 0$).

In turn, the partial derivatives of the log-likelihood function L_n, with respect to the parameters c and α, are given by

$$\frac{\partial L_n}{\partial c}(t_1, \ldots, t_n; c, \alpha) = n\frac{1}{c} - (t_n)^{-\alpha} \tag{37}$$

and

$$\frac{\partial L_n}{\partial \alpha}(t_1, \ldots, t_n; c, \alpha) = n\frac{1}{\alpha} - \ln(t_1 \cdots t_n) + c\,(t_n)^{-\alpha} \ln(t_n). \tag{38}$$

The log-likelihood function L_n has a unique critical point (c_*, α_*). Indeed, equating equation (37) to zero implies that $n = c_*(t_n)^{-\alpha_*}$; substituting this into (38), while equating it to zero, further implies that

$$\frac{1}{\alpha_*} = \frac{1}{n}\ln(t_1\cdots t_n) - \ln(t_n) = \frac{1}{n}\sum_{k=1}^{n-1}\ln\left(\frac{t_k}{t_n}\right). \tag{39}$$

The critical point (c_*, α_*) is a global maximum of the log-likelihood function L_n. Hence, (39) implies that the statistic n/Θ_n is a Maximal Likelihood Estimator of the Paretian exponent α.

9.3.2 The Distribution of the Statistic Θ_n

The stochastic representation of (14) implies that

$$\ln\left(\frac{O_k}{O_n}\right) = \ln\left(\frac{O_k}{O_{k+1}}\cdots\frac{O_{n-1}}{O_n}\right)$$
$$\overset{\text{law}}{=} -\ln\left((U_k)^{\frac{1}{\alpha k}}\cdots(U_n)^{\frac{1}{\alpha(n-1)}}\right) = \frac{-1}{\alpha}\sum_{j=k}^{n-1}\frac{1}{j}\ln(U_j) \tag{40}$$

$(k = 1, \ldots, n-1)$. Setting $Z_j = -\ln(U_j)$ $(j = 1, \ldots, n-1)$ we further obtain that

$$\Theta_n = \sum_{k=1}^{n-1}\ln\left(\frac{O_k}{O_n}\right)$$
$$\overset{\text{law}}{=} \sum_{k=1}^{n-1}\frac{1}{\alpha}\sum_{j=k}^{n-1}\frac{1}{j}Z_j = \sum_{j=1}^{n-1}\frac{1}{j}Z_j\sum_{k=1}^{j}\frac{1}{\alpha}, \tag{41}$$
$$\sum_{j=1}^{n-1}\frac{1}{j}Z_j\frac{j}{\alpha} = \frac{1}{\alpha}(Z_1 + \cdots + Z_{n-1}).$$

Since the random variables $\{U_1, \ldots, U_{n-1}\}$ are IID and uniformly distributed on the unit interval, the random variables $\{Z_1, \ldots, Z_{n-1}\}$ are IID and exponentially distributed with unit mean.

The stochastic representation of (41) implies that the statistic Θ_n is Gamma distributed with parameters α and $n-1$. Namely, the statistic Θ_n is governed by the probability density function

$$f(t) = \frac{\alpha^{n-1}}{(n-2)!}\exp\{-\alpha t\}\,t^{n-2} \quad (t > 0). \tag{42}$$

9.3.3 Statistical Properties of the Estimators

Let η be an arbitrary positive parameter and consider the random variable η/Θ_n. Equation (42) implies that the random variable η/Θ_n is governed by the probability density function

$$f(t) = \frac{(\alpha\eta)^{n-1}}{(n-2)!}\exp\{-\alpha\eta/t\}\,t^{-n} \quad (t > 0). \tag{43}$$

The probability density function of (43):

1. is unimodal, and its mode and peak are given, respectively, by

$$t_* = \frac{\alpha\eta}{n} \quad \text{and} \quad f(t_*) = \frac{1}{\alpha\eta}\frac{n^n \exp\{-n\}}{(n-2)!}; \tag{44}$$

2. has mean

$$\frac{\alpha\eta}{n-2}; \tag{45}$$

3. has variance

$$\left(\frac{\alpha\eta}{n-2}\right)^2 \frac{1}{n-3}; \tag{46}$$

4. has skew

$$4\frac{\sqrt{n-3}}{n-4}; \tag{47}$$

5. has kurtosis

$$3\frac{(n-3)(n+4)}{(n-4)(n-5)}. \tag{48}$$

Setting $\eta = n$, and using Stirling's formula, (44) implies that the statistic n/Θ_n has mode α and peak $\sim \sqrt{n}/(\alpha\sqrt{2\pi})$ (as $n \to \infty$). Setting $\eta = n-2$, (45)–(46) imply that the statistic $(n-2)/\Theta_n$ has mean α and variance $\alpha^2/(n-3)$. Last, (47)–(48) give the skew and the kurtosis of both the statistics n/Θ_n and $(n-2)/\Theta_n$.

9.4 Poissonian Renormalization: Motivation

Consider the renormalization parameter p to be an *integer*, and set the Poisson processes $\Pi^{(1)}, \dots, \Pi^{(p)}$ to be IID copies of the Poisson process Π.

Since p is an integer, the Poisson process Π_p—governed by the rate function $\mathbf{r}_p(t) = p\mathbf{r}(t)$—is the union (superposition) of p independent copies of the Poisson process Π [5]. Namely: $\Pi_p = \Pi^{(1)} \cup \dots \cup \Pi^{(p)}$. Hence, the p-order renormalization $\mathcal{R}_p(\Pi)$ is given by

$$\mathcal{R}_p(\Pi) = \left\{\frac{1}{p^\beta} \cdot \pi\right\}_{\pi \in \Pi^{(1)} \cup \dots \cup \Pi^{(p)}}. \tag{49}$$

In particular, the maximum $M_{\mathcal{R}_p(\Pi)}$ of the p-order renormalization $\mathcal{R}_p(\Pi)$ is given by

$$M_{\mathcal{R}_p(\Pi)} = \frac{\max\{M_{\Pi^{(1)}}, \dots, M_{\Pi^{(p)}}\}}{p^\beta} \tag{50}$$

(the random variables $M_{\Pi^{(1)}}, \dots, M_{\Pi^{(p)}}$ being, respectively, the maxima of the Poisson processes $\Pi^{(1)}, \dots, \Pi^{(p)}$). And, the aggregate $A_{\mathcal{R}_p(\Pi)}$ of the p-order renormalization $\mathcal{R}_p(\Pi)$ is given by

$$A_{\mathcal{R}_p(\Pi)} = \frac{A_{\Pi^{(1)}} + \dots + A_{\Pi^{(p)}}}{p^\beta} \tag{51}$$

(the random variables $A_{\Pi^{(1)}}, \dots, A_{\Pi^{(p)}}$ being, respectively, the aggregates of the Poisson processes $\Pi^{(1)}, \dots, \Pi^{(p)}$).

Extreme Value Theory seeks the fixed points of (50) [9]–[11]: A positive-valued random variable M that satisfies

$$M \overset{\text{law}}{=} \frac{\max\{M_1, \ldots, M_p\}}{p^{\beta}}, \tag{52}$$

where $\{M_k\}_{k=1}^{\infty}$ are IID copies of M. The fixed points of (52) turn out to be *Fréchet distributed* ((6) with exponent $\alpha = 1/\beta$).

The Central Limit Theorem seeks the fixed points of (51) [12]–[14]: A positive-valued random variable A that satisfies

$$A \overset{\text{law}}{=} \frac{A_1 + \cdots + A_p}{p^{\beta}}, \tag{53}$$

where $\{A_k\}_{k=1}^{\infty}$ are IID copies of A. The fixed points of (53) turn out to be one-sided *Lévy distributed* ((7) with exponent $\alpha = 1/\beta$).

Transcending from the maximum and the aggregate projections back up to the 'process level', we seek the fixed points of (49): A Poisson process Π, defined on the positive half-line, that satisfies

$$\mathcal{R}_p(\Pi) \overset{\text{law}}{=} \Pi. \tag{54}$$

The Poissonian renormalization $\mathcal{R}_p(\Pi)$ is thus the infinite-dimensional 'meta' renormalization underlying the one-dimensional renormalizations of both Extreme Value Theory (52) and the Central Limit Theorem (53) (in the case of positive-valued IID random variables). Moreover, the Poissonian renormalization $\mathcal{R}_p(\Pi)$ is naturally extended from integer-valued parametrization ($p = 1, 2, \ldots$) to positive-valued parametrization ($p > 0$)—as presented in Sect. 6.2.

9.5 Proof of Proposition 6

If the Poisson process Π is Paretian then (22) implies that it is resilient. We assume that Π is a Poisson process with rate function $\mathbf{r}(t)$ ($t > 0$) which is integrable at infinity, and which satisfies (23), and prove that it is Paretian.

Let p and q be arbitrary positive parameters. Clearly

$$\Upsilon_{pq}(\Pi) = \Upsilon_p\left(\Upsilon_q(\Pi)\right) \tag{55}$$

(this identity holds for all Poisson processes). Equation (55), combined together with (23), implies that

$$\Lambda(pq)\mathbf{r}(t) = \left(\Upsilon_{pq}(\mathbf{r})\right)(t) = \left(\Upsilon_p\left(\Upsilon_q(\mathbf{r})\right)\right)(t) = \Lambda(p)\Lambda(q)\mathbf{r}(t) \tag{56}$$

($t > 0$). Hence $\Lambda(pq) = \Lambda(p)\Lambda(q)$ which, in turn, implies that $\Lambda(p) = p^{\gamma}$ where γ is an arbitrary real-valued exponent.

Now, (21) implies that

$$\left(\Upsilon_{1/p}(\mathbf{r})\right)(t) = p\mathbf{r}(pt) \quad (t > 0). \tag{57}$$

On the other hand, (23) implies that

$$\left(\Upsilon_{1/p}(\mathbf{r})\right)(t) = p^{-\gamma}\mathbf{r}(t) \quad (t > 0). \tag{58}$$

Consequently, (57) and (58) yield

$$\mathbf{r}(pt) = p^{-(1+\gamma)}\mathbf{r}(t) \quad (t > 0). \tag{59}$$

Finally, setting $t = 1$ in (59) and using the fact that the parameter p is arbitrary, we conclude that: (i)

$$\mathbf{r}(p) = \mathbf{r}(1)p^{-(1+\gamma)} \quad (p > 0); \tag{60}$$

and, (ii) the exponent γ is positive (for otherwise the rate function will fail to be integrable at infinity). Hence, we obtained that the Poisson process Π is Paretian.

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